Cauchy Distribution

Let 'X' be a continuous random variable with interval (-∞,∞) is said to be Cauchy distribution, having p.d.f:

$$f(x)=\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right] \qquad -\infty\le x\le\infty$$

It has two parameters λ *and* α..

If $\lambda=1$ then it becomes single parameter Cauchy distribution with p.d.f

$$f(x)=\frac{1}{\pi}\left[\frac{1}{1+(x-\alpha)^2}\right] \qquad -\infty\le x\le\infty$$

If $\alpha=0$ *and* $\lambda=1$ then it becomes standard Cauchy distribution with p.d.f

$$f(x)=\frac{1}{\pi}\left[\frac{1}{1+x^2}\right] \qquad or \qquad \frac{1}{\pi}\left[\frac{1}{1+z^2}\right] \qquad -\infty\le x\le\infty$$

Where α = notation parameter. λ = scale parameter.

Properties:

i) Cauchy distribution is a continuous distribution.

ii) The total area under the curve is unity.

iii) The range of the distribution is -∞ to ∞.

iv) It has two parameters α & λ..

v) The mean of the Cauchy distribution does not exist.

vi) The variance of the Cauchy distribution does not exist.

vii) The mode of Cauchy distribution is $\hat{x}=\alpha$

viii) The median of Cauchy distribution is $m=\alpha$

Prove that total area under the curve is unity

Solution: Let by definition

Total Area= $\int_{-\infty}^{\infty} f(x)dx$

As x ≈ Cauchy (λ,α) with p.d.f

$$f(x)=\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right] \qquad -\infty\le x\le\infty$$

Area=

$$\int_{-\infty}^{\infty}\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right]dx=\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\lambda^2+(x-\alpha)^2}\right]dx=\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\lambda^2\left(1+(\frac{x-\alpha}{\lambda})^2\right)}\right]dx$$

Put $z=\dfrac{x-\alpha}{\lambda}, \qquad x-\alpha=z\lambda, \qquad x=\alpha+z\lambda, \qquad dx=\lambda dz$

Limits remain same.

$$\text{Area}=\frac{\lambda}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{\lambda^2(1+z^2)}\right]\lambda dz=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(1+z^2)}\right]dz$$

$$\text{Area}=\frac{2}{\pi}\int_{0}^{\infty}\left[\frac{1}{(1+z^2)}\right]dz \qquad \text{It is an even function. So}$$

$$\text{Area}=\frac{2}{\pi}\tan^{-1}(z)\Big|_0^{\infty}=\frac{2}{\pi}\left[\tan^{-1}(\infty)-\tan^{-1}(0)\right]$$

Therefore $\tan^{-1}(\infty)=\dfrac{\pi}{2}$

$$\text{Area} = \frac{2}{\pi}\left[\tan^{-1}(\infty) - \tan^{-1}(0)\right] = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1$$

Find mean of Cauchy distribution

Solution: Let by definition

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$

As x ≈ Cauchy (λ, α) with p.d.f

$$f(x) = \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right] \qquad -\infty \le x \le \infty$$

$$E(x) = \int_{-\infty}^{\infty} x \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right]dx = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} x\left[\frac{1}{\lambda^2 + (x-\alpha)^2}\right]dx = \frac{\lambda}{\pi}\int_{-\infty}^{\infty} x\left[\frac{1}{\lambda^2\left(1 + (\frac{x-\alpha}{\lambda})^2\right)}\right]dx$$

Put $z = \frac{x-\alpha}{\lambda}, \qquad x - \alpha = z\lambda, \qquad x = \alpha + z\lambda, \qquad dx = \lambda dz$

Limits remain same.

$$E(x) = \frac{\lambda}{\pi}\int_{-\infty}^{\infty}\left[\frac{(\alpha + z\lambda)}{\lambda^2(1+z^2)}\right]\lambda dz = \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{(\alpha + z\lambda)}{(1+z^2)}\right]dz = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+z^2)}dz + \frac{\lambda}{\pi}\int_{-\infty}^{\infty}\frac{z}{(1+z^2)}dz$$

$$E(x) = \frac{\alpha}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+z^2)}dz + \frac{\lambda}{2\pi}\int_{-\infty}^{\infty}\frac{2z}{(1+z^2)}dz = \frac{2\alpha}{\pi}\int_{0}^{\infty}\frac{1}{(1+z^2)}dz + \frac{\lambda}{2\pi}\log(1+z^2)\Big]_{-\infty}^{\infty}$$

$$E(x) = \frac{2\alpha}{\pi}\int_{0}^{\infty}\frac{1}{(1+z^2)}dz + \frac{\lambda}{2\pi}\log(1+\infty^2) - \log(1+\infty^2) = \frac{2\alpha}{\pi}\int_{0}^{\infty}\frac{1}{(1+z^2)}dz + \frac{\lambda}{2\pi}(\infty) = \infty$$

Hence, prove that the mean of Cauchy distribution does not exist.

Find median of Cauchy distribution

Let by definition of median

$$P(x < m) = \frac{1}{2}$$

As x ≈ Cauchy (λ, α) with p.d.f

$$f(x) = \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right] \qquad -\infty \le x \le \infty$$

$$\int_{-\infty}^{m}\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right]dx = \frac{1}{2}$$

$$\frac{\lambda}{\pi}\int_{-\infty}^{m}\left[\frac{1}{\lambda^2 + (x-\alpha)^2}\right]dx = \frac{1}{2}$$

$$\frac{\lambda}{\pi}\int_{-\infty}^{m}\left[\frac{1}{\lambda^2\left(1 + (\frac{x-\alpha}{\lambda})^2\right)}\right]dx = \frac{1}{2}$$

Put $z = \frac{x-\alpha}{\lambda}, \qquad x - \alpha = z\lambda, \qquad x = \alpha + z\lambda, \qquad dx = \lambda dz$

While limits will be when $x \to -\infty \, then \, z \to -\infty$, when $x \to m \, then \, z \to \frac{m-\alpha}{\lambda}$

$$\frac{\lambda}{\pi}\int_{-\infty}^{\frac{m-\alpha}{\lambda}}\left[\frac{1}{\lambda^2(1+z^2)}\right]\lambda dz = \frac{1}{\pi}\int_{-\infty}^{\frac{m-\alpha}{\lambda}}\left[\frac{1}{(1+z^2)}\right]dz = \frac{1}{2} \qquad \text{(i)}$$

As, we know that

$$\int_{-\infty}^{\infty} f(z)dz = 1$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{(1+z^2)}\right]dz = 1$$

$$\frac{1}{\pi}\int_{-\infty}^{0}\left[\frac{1}{(1+z^2)}\right]dz = \frac{1}{\pi}\int_{0}^{\infty}\left[\frac{1}{(1+z^2)}\right]dz = \frac{1}{2}$$

$$\frac{1}{\pi}\int_{-\infty}^{0}\left[\frac{1}{(1+z^2)}\right]dz = \frac{1}{2} \qquad \text{(ii)}$$

Comparing (i) & (ii)

$\frac{m-\alpha}{\lambda} = 0, \qquad m-\alpha = 0, \qquad m = \alpha$ Require median.

Find Cumulative Density Function of Cauchy distribution or Find the Distribution Function.

Solution: Let by definition

$$F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)dx$$

As x $\approx$ Cauchy (λ, α) with p.d.f

$$f(x) = \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right] \qquad -\infty \le x \le \infty$$

$$F(x) = \int_{-\infty}^{x}\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right]dx = \frac{\lambda}{\pi}\int_{-\infty}^{x}\left[\frac{1}{\lambda^2+(x-\alpha)^2}\right]dx = \frac{\lambda}{\pi}\int_{-\infty}^{x}\left[\frac{1}{\lambda^2\left(1+(\frac{x-\alpha}{\lambda})^2\right)}\right]dx$$

Put $z = \frac{x-\alpha}{\lambda}, \qquad x-\alpha = z\lambda, \qquad x = \alpha + z\lambda, \qquad dx = \lambda dz$

While limits will be when $x \to -\infty \, then \, z \to -\infty$, when $X \to x \, then \, z \to \frac{x-\alpha}{\lambda}$

$$\frac{\lambda}{\pi}\int_{-\infty}^{\frac{x-\alpha}{\lambda}}\left[\frac{1}{\lambda^2(1+z^2)}\right]\lambda dz = \frac{1}{\pi}\int_{-\infty}^{\frac{x-\alpha}{\lambda}}\left[\frac{1}{(1+z^2)}\right]dz = \frac{1}{\pi}\tan^{-1}(z)\Big|_{-\infty}^{\frac{x-\alpha}{\lambda}} = \frac{2}{\pi}\left[\tan^{-1}(\frac{x-\alpha}{\lambda}) - \tan^{-1}(-\infty)\right]$$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1}(\frac{x-\alpha}{\lambda}) - (-\frac{\pi}{2})\right] = \frac{1}{\pi}\left[\tan^{-1}(\frac{x-\alpha}{\lambda}) + (\frac{\pi}{2})\right] = \frac{1}{\pi}\tan^{-1}(\frac{x-\alpha}{\lambda}) + \frac{1}{\pi}\frac{\pi}{2}$$

$$F(x) = \frac{1}{\pi}\tan^{-1}(\frac{x-\alpha}{\lambda}) + \frac{1}{2}$$ Require result.

Find Characteristic Function of Cauchy distribution

Solution: Let by definition of characteristic function

$$\theta_x(t) = E(e^{itx}) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$$

$$f(x) = \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2 + (x-\alpha)^2}\right] \qquad -\infty \le x \le \infty$$

$$\theta_x(t) = \int_{-\infty}^{\infty} e^{itx}\frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right]dx$$

$$\theta_x(t)=\frac{\lambda}{\pi}\int_{-\infty}^{\infty}e^{itx}\left[\frac{1}{\lambda^2\left(1+(\frac{x-\alpha}{\lambda})^2\right)}\right]dx$$

Put $z=\frac{x-\alpha}{\lambda},\ x-\alpha=z\lambda,\ x=\alpha+z\lambda,\ \ dx=\lambda dz$ Limits remain same.

$$\theta_x(t)=\frac{\lambda}{\pi}\int_{-\infty}^{\infty}e^{it(\alpha+\lambda z)}\left[\frac{1}{\lambda^2\left(1+z^2\right)}\right]\lambda dz$$

$$\theta_x(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left[\frac{e^{it(\alpha+\lambda z)}}{\left(1+z^2\right)}\right]dz$$

$$\theta_x(t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\alpha+it\lambda z}}{\left(1+z^2\right)}dz$$

$$\theta_x(t)=e^{it\alpha}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\lambda z}}{\left(1+z^2\right)}dz \qquad \text{(A)}$$

Now consider $=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\lambda z}}{\left(1+z^2\right)}dz$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\lambda z}}{\left(1+z^2\right)}dz \qquad put\,t'=t\lambda$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it\lambda z}}{\left(1+z^2\right)}dz=\int_{-\infty}^{\infty}\frac{e^{it'z}}{\left(1+z^2\right)}dz \qquad \text{(i)}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{it'z}}{\left(1+z^2\right)}dz=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Cos(t'z)+iSin(t'z)}{1+z^2}dz=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Cos(t'z)}{1+z^2}dz+\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{Sin(t'z)}{1+z^2}dz$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{Cos\left(t'z\right)}{1+z^2}dz \qquad \text{(ii)}$$

skip the next function because it is an odd function.

By inverse theorem

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}dz\vartheta_x(t)dt$$

As we know that the p.d.f of Laplace distribution

$$f(x)=\frac{1}{2}e^{-|x|}$$

And we know that the characteristic function of laplace distribution

$$\theta_x(t)=\frac{1}{\left(1+t^2\right)}$$

$$\frac{1}{2}e^{-|x|}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}\frac{1}{\left(1+t^2\right)}dt$$

$$e^{-|x|}=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-itx}\frac{1}{\left(1+t^2\right)}dt$$ Replacing‘t’ by ‘x’ & ‘x’ by‘t’

$$e^{-|t'|}=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-it'x}\frac{1}{\left(1+x^2\right)}d$$ And replace ‘x’ by ‘z’

$$e^{-|t'|}=\frac{1}{\pi}\int_{-\infty}^{\infty}e^{-it'z}\frac{1}{\left(1+z^2\right)}dz$$

$$e^{-|t'|}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Cos\left(t'z\right)-iSin\left(t'z\right)}{1+z^2}dz$$

$$e^{-|t'|} = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{Cos(t'z)}{1+z^2}dz - \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{Sin(t'z)}{1+z^2}dz$$

$$e^{-|t'|} = \frac{2}{\pi}\int_{0}^{\infty}\frac{Cos(t'z)}{1+z^2}dz$$ Put in (i)

$$\int_{-\infty}^{\infty}\frac{e^{it'z}}{(1+z^2)}dz = e^{-|t'|}$$ and $put\, t' = t\lambda$ put in (A)

$$\theta_x(t) = e^{it\alpha}e^{-|t\lambda|}$$

$$\theta_x(t) = e^{it\alpha-|t\lambda|}$$

In standard form $\alpha = 0$ and $\lambda = 1$

$$\vartheta_x(t) = e^{-|t|}$$

Find the mode of Cauchy distribution.

If following two conditions are satisfied then mode exists.

$f(x') = 0$ or $\frac{d}{dx}\log f(x) = 0$

$f(x') < 0$ or $\frac{d^2}{dx^2}\log f(x) < 0$

As x ≈ Cauchy (λ, α) with p.d.f

$$f(x) = \frac{1}{\pi}\left[\frac{\lambda}{\lambda^2+(x-\alpha)^2}\right] \qquad -\infty \le x \le \infty$$

$$f(x) = \frac{\lambda}{\pi}\left(\lambda^2+(x-\alpha)^2\right)^{-1}$$

$$\frac{d}{dx}f(x) = \left[\frac{\lambda}{\pi} - 1\left(\lambda^2+(x-\alpha)^2\right)^{-2}\frac{d}{dx}\left(\lambda^2+(x-\alpha)^2\right)\right] = \left[-\frac{\lambda}{\pi}\left(\lambda^2+(x-\alpha)^2\right)^{-2}\left(2(x-\alpha)\right)\right] \text{(i)}$$

Put equal to zero.

$$\left[-\frac{\lambda}{\pi}\left(\lambda^2+(x-\alpha)^2\right)^{-2}\left(2(x-\alpha)\right)\right] = 0, \qquad \frac{x-\alpha}{\left(\lambda^2+(x-\alpha)^2\right)^{-2}} = 0, \quad x-\alpha = 0, \qquad x = \alpha$$

Again differentiate (i) w.r.t to 'x'

$$\frac{d^2}{dx^2}f(x) = \frac{d}{dx}\left[-\frac{\lambda}{\pi}\left(\lambda^2+(x-\alpha)^2\right)^{-2}\left(2(x-\alpha)\right)\right]$$

$$\frac{d^2}{dx^2}f(x) = -\frac{2\lambda}{\pi}\left[\left(\lambda^2+(x-\alpha)^2\right)^{-2}\frac{d}{dx}(x-\alpha) + (x-\alpha)\frac{d}{dx}\left(\lambda^2+(x-\alpha)^2\right)^{-2}\right]$$

$$\frac{d^2}{dx^2}f(x) = -\frac{2\lambda}{\pi}\left[\left(\lambda^2+(x-\alpha)^2\right)^{-2}(1) + (x-\alpha) - 2\left(\lambda^2+(x-\alpha)^2\right)^{-3}\frac{d}{dx}\left(\lambda^2+(x-\alpha)^2\right)\right]$$

$$\frac{d^2}{dx^2}f(x) = -\frac{2\lambda}{\pi}\left[\left(\lambda^2+(x-\alpha)^2\right)^{-2} + (x-\alpha) - 2\left(\lambda^2+(x-\alpha)^2\right)^{-3}2(x-\alpha)\right]$$

$$\frac{d^2}{dx^2}f(x) = -\frac{2\lambda}{\pi}\left[\left(\lambda^2+(x-\alpha)^2\right)^{-2} - 4(x-\alpha)^2\left(\lambda^2+(x-\alpha)^2\right)^{-3}\right]$$

Put $x = \alpha$

$$\frac{d^2}{dx^2}f(x)=-\frac{2\lambda}{\pi}\left[\left(\lambda^2+(\alpha-\alpha)^2\right)^{-2}-4(\alpha-\alpha)^2\left(\lambda^2+(\alpha-\alpha)^2\right)^{-3}\right]$$

$$\frac{d^2}{dx^2}f(x)=-\frac{2\lambda}{\pi}\left[\frac{1}{\left(\lambda^2\right)^2}+0\right]$$

$$\frac{d^2}{dx^2}f(x)=-\frac{2}{\pi\lambda^3}<0$$

Which shows that the mode of Cauchy Distribution is $=\hat{x}=\alpha$

Mode of Cauchy Distribution in standard form

If following two conditions are satisfied then mode exists.

$f(x')=0$ or $\frac{d}{dx}\log f(x)=0$

$f(x')<0$ or $\frac{d^2}{dx^2}\log f(x)<0$

As x ≈ Cauchy ($0,1$) with p.d.f

$$f(x)=\frac{1}{\pi}\left[\frac{1}{1+x^2}\right] \qquad -\infty\le x\le\infty$$

$$f(x)=\frac{1}{\pi}\left(1+x^2\right)^{-1}$$

Differentiate w.r.t. to ‘x’

$$\frac{d}{dx}f(x)=\left[\frac{1}{\pi}-1\left(1+x^2\right)^{-2}\frac{d}{dx}\left(1+x^2\right)\right]$$

$$\frac{d}{dx}f(x)=\left[-\frac{1}{\pi}\left(1+x^2\right)^{-2}2x\right]$$

$$\frac{d}{dx}f(x)=-\frac{2x}{\pi}\left(1+x^2\right)^{-2} \qquad \text{(i)}$$

Put equal to zero.

$$-\frac{2x}{\pi}\left(1+x^2\right)^{-2}=0, \qquad -\frac{2x}{\pi\left(1+x^2\right)^2}=0, \qquad 2x=0$$

$$x=0$$

Again differentiate (i) w.r.t to ‘x’

$$\frac{d^2}{dx^2}f(x)=\frac{d}{dx}\left[-\frac{2x}{\pi}\left(1+x^2\right)^{-2}\right]$$

$$\frac{d^2}{dx^2}f(x)=\left[-\frac{2x}{\pi}\frac{d}{dx}\left(1+x^2\right)^{-2}+\left(1+x^2\right)^{-2}\frac{d}{dx}\left(-\frac{2x}{\pi}\right)\right]$$

$$\frac{d^2}{dx^2}f(x)=-\frac{2x}{\pi}(-2)\left(1+x^2\right)^{-3}(2x)+\left(1+x^2\right)^{-2}\left(-\frac{2}{\pi}\right)$$

Put x=0

$$\frac{d^2}{dx^2}f(x)=\frac{\theta}{\pi\left(1+(0)^2\right)^3}-\frac{2}{\pi\left(1+(0)^2\right)^2}$$

$$\frac{d^2}{dx^2} f(x) = 0 - \frac{2}{\pi} = -\frac{2}{\pi} < 0$$

As the both conditions are satisfied hence the mode of Cauchy distribution is $\hat{x} = 0$

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